BRIEF REPORTS

Brief Reports are accounts of completed research which do not warrant regular articles or the priority handling given to Rapid Communications; however, the same standards of scientific quality apply. (Addenda are included in Brief Reports.) A Brief Report may be no longer than 4 printed pages and must be accompanied by an abstract. The same publication schedule as for regular articles is followed, and page proofs are sent to authors.

Synchronization of coupled maps and stable windows

Troy Shinbrot
Laboratory for Fluid Mechanics, Chaos, and Mixing, Department of Chemical Engineering, Northwestern University, Evanston, Illinois 60208
(Received 14 January 1994)

Synchronization among globally coupled, chaotic map lattices can be related to stable periodic windows in isolated chaotic maps. This relation provides a simple predictive tool for the understanding of complicated behavior in coupled systems.

PACS number(s): 05.45.+b, 05.90.+m, 87.10.+e

Synchronization of coupled map lattices (CML's) has been studied by several authors in recent years [1,2]. CML's can be viewed as an intermediate state between low dimensional dynamical behavior, seen in simple maps, and infinite dimensional behavior, for example, of turbulent fluids. Moreover, understanding of synchronization of coupled systems is important for the study of a variety of physical [3], chemical [4], and biological [5] problems. Under some conditions, all of the maps in an ensemble will spontaneously synchronize and behave as one. Under slightly different conditions, they will break into two, three, or many distinct clusters. And under yet different conditions, the maps will behave nearly independently. In this Brief Report, we address several questions. Why should coupled maps behave in such a complicated way? What is the mechanism for synchronization? Is there a relation between the nonlinearity in component maps and synchronization, or could linear maps also synchronize? To address these questions, we describe one mechanism which leads to synchronization in globally coupled maps.

Let us begin by defining a globally coupled CML composed of N identical sites:

$$X_{n+1}(i)=F(X_n(i))+\frac{\kappa}{N-1}\sum_{j=1;j\neq i}^{N}X_n(j)\ , \tag{1}$$

where $X_n(i)$ denotes the state of the ith site at the nth iterate. The map $F(X)$ then defines the evolution of the state of a single isolated site, and the constant κ determines the strength of the coupling between this site and all of the other sites in the lattice. For simplicity, we take X to be a scalar, and we study coupled logistic maps, $F(X)=\alpha X(1-X)$, where we choose the constant α to lie in the chaotic regime of the map. Rather than trying to decipher the coupling expression, which is quite complicated, we will focus on the behavior of an individual map site, which is much simpler.

The crucial observation of this Brief Report is that individual maps frequently exhibit periodic [6] and *stable* windows. That is, even if an isolated map is specified to lie in the chaotic regime, the coupling parameter may often be such as to perturb the map to an attractive, e.g., periodic, state. For example, let $\alpha=3.8$, which is in the chaotic regime. This map, defined by the state X_n, can be brought into a large period-3 window by the addition of a constant term c,

$$X_{n+1}=3.8X_n(1-X_n)+c\ , \tag{2}$$

where c is in the interval $I=[0.011,0.015]$. This is demonstrated in Fig. 1, which shows a bifurcation diagram for the map, Eq. (2). For each value of c in the figure, we iterated Eq. (2) 1000 times starting from the initial state $X_0=0.5$ and then plotted the values of X_{1001} through X_{1064}.

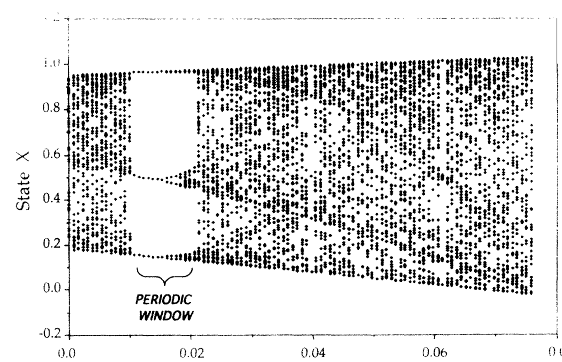

FIG. 1. Bifurcation diagram for Eq. (2).

1063-651X/94/50(4)/3230(4)/$06.00

The presence of a periodic window for an *isolated* map implies that whenever the coupling term in the *coupled* maps lies in I, we can expect all initial conditions on [0,1] to converge onto one of the periodic points. This effect can be demonstrated by choosing κ in Eq. (1) so that the coupling term for given initial conditions is in the desired range. Suppose, for example, that we choose a uniform initial distribution of states $X_n(0)$. This distribution rapidly approaches an asymptotic distribution with mean $\bar{X} \cong 0.64$. In order to bring the maps to the periodic window, we require that Eqs. (1) and (2) agree, i.e., we require $\kappa 0.64 = c$. Now if we choose c to lie in I, say $c = 0.015$, then we obtain $\kappa \cong 0.023$. This suggests that if we set κ to be 0.023, then the states of a CML may converge to the periodic points.

The completion of this migration depends on two things. First, the actual average state of the maps can be expected to vary from the calculated value $\bar{X}$. The actual average $\bar{X}_a$ will change from iterate to iterate even in the isolated map, but in addition, the average state in the synchronized condition $\bar{X}_s$, will in general differ from the average state for the isolated map. Nevertheless, for a sufficiently large stable window, such as the one shown in Fig. 1, both $k_a = c/\bar{X}_a$ and $k_s = c/\bar{X}_s$ can be contained in the window and synchronization can be observed. In particular, we note that for our example, every time that the mean map state approaches 0.64 (which can be expected to occur often) the synchronized period-three state will be reinforced. Second, the convergent influence of the periodic window may or may not exceed the divergent influence of the chaotic map. Thus whether synchronization actually occurs or not depends on whether the inverse of the Lyapunov exponent of the chaotic map exceeds the characteristic convergence time within the periodic window. If so, synchronization can occur; otherwise it cannot.

Figure 2 shows synchronization to the period-three state for the example described above. In the figure, we show the time evolution (in five-iterate time steps for clarity) of an initially uniform distribution of 100 lattice sites. Note that all of the sites are displayed on every fifth time step. Within 500 iterates, all of the sites are clustered [7] very near to the period-three state, indicated by the three arrows in the figure. Identical results have been obtained using 1000 and 10 000 sites, with the exception that synchronization occurs more rapidly with more sites.

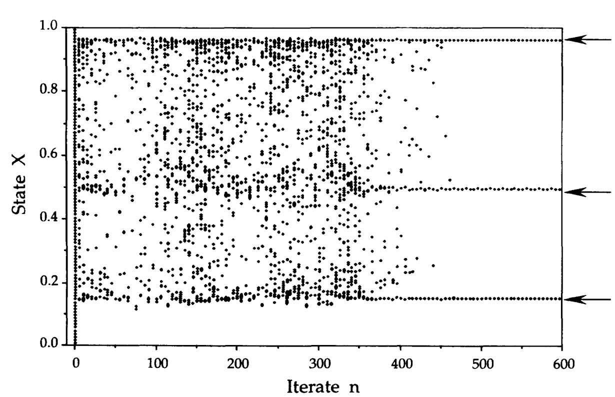

FIG. 2. Synchronization of 100 globally coupled logistic maps ($k = 0.023$ to hit $c = 0.015$); we mention that *all* map states are plotted.

Apparently, we can exploit the mechanism described above to rationalize synchronization in coupled chaotic maps. Moreover, we can use knowledge of this mechanism to predict in advance when synchronization will occur. For example, for the logistic maps discussed above, the entire periodic window shown in Fig. 1 extends from $c \cong 0.011$ to $c \cong 0.021$. The asymptotic mean of the isolated logistic map is $\bar{X} \cong 0.64$, while the mean of the synchronized periodic state is about $\bar{X}_s \cong 0.53$. The stable window must persist for both the isolated logistic map and for the synchronized state, so we require both $\kappa \bar{X}_a$ and $\kappa \bar{X}_s$ to lie within (0.011,0.021). Thus κ must lie in (0.022,0.035). In Fig. 3, we show a confirmation of this prediction. In the figure, we show the results of iterating 100 coupled maps 1000 times starting from a uniform distribution of initial map states. For each value of κ, we display the 1001st iterate for all 100 maps simultaneously. As expected, there is a large synchronized window extending from about $\kappa \cong 0.022$ to $\kappa \cong 0.035$.

The example shown above is particularly simple, because only one large periodic window is evident in Fig. 1. By modifying the form of the coupling [8], however, one can generate other large stable regions and hence other possible synchronization patterns. For example,one CML that has been studied recently [1] is of the form

$$X_{n+1}(i) = (1-\kappa)F(X_n(i)) + \frac{\kappa}{N}\sum_{j=1}^{N} F(X_n(j)) \ . \tag{3}$$

The presence of the coupling constant κ in two terms confounds the problem; nevertheless, we can adopt essentially the same strategy as before. An isolated map is defined by

$$X_{n+1} = (1-\kappa)F(X_n) + \kappa c' \ , \tag{4}$$

where c' is a constant which determines the average influence of the ensemble of maps. We can approximately determine the value of the additive term in Eq. (4) by observing that c' is $O(1)$ and is never far from 0.5. If we use the approximation $c' \cong 0.5$ and retain the previous form for $F(X)$ [i.e., $F(X) = 3.8X(1-X)$], then by applying precisely the procedure described for Fig. 1, we ob-

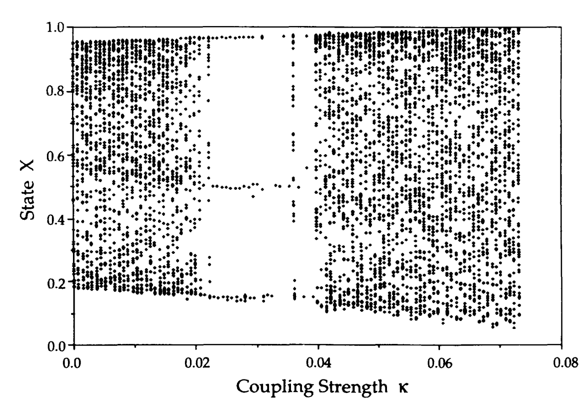

FIG. 3. Synchronization diagram for logistic map lattice.

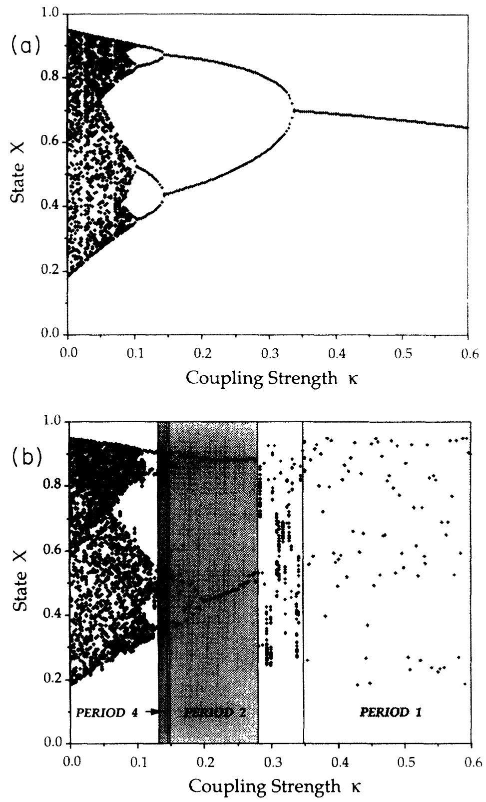

FIG. 4. (a) Bifurcation diagram for isolated logistic map, Eq. (4); (b) synchronization diagram for 100 coupled logistic maps, Eq. (3).

tain the bifurcation diagram shown in Fig. 4(a). In the figure, a variety of stable states are evident, extending all of the way back through the inverse period doubling cascade to a stable fixed point. Correspondingly, we find an inverse synchronization cascade, shown in Fig. 4(b) where we plot the 1001st iterate of 100 initially uniform coupled maps. Note that all of the maps in the rightmost grey region in Fig. 4(b) are synchronized to a single X value. The values of X shown do not follow an obvious pattern as κ is changed, and in our numerical experiments we have found that these values depend on the initial state of the maps. This is to be expected for chaotic maps. Moreover, synchronization is not predictable in the bands close to the bifurcations between periods, 1, 2, and 4. This, too, is to be expected since the location of the bifurcations will vary with the chosen value of c', which is only known approximately. The point to be stressed is that the correspondence between Figs. 4(a) and 4(b) is unmistakable, and by the 1001st iterate all 100 maps have synchronized to one, two, or four values of X in each of the three grey shaded regions of Fig. 4(b).

We can now provide answers to the questions posed at the beginning of this Brief Report. First, coupled maps can be expected to synchronize in complicated ways when more than one stable state is present in the isolated map. By studying the stable states of the isolated map, we can deduce some of the synchronization behavior of the coupled system. Second, we have presented one possible mechanism that leads to synchronization. Other mechanisms may exist also. Irrespective of the details of these mechanisms, to be effective they must share the property that the synchronized state is stable. For example,it is informative to rewrite Eq. (1) in the following form:

$$X_{n+1}(i)=F(X_n(i))+\kappa X_n(i) + \frac{\kappa}{N-1}\sum_{j=1;j\neq i}^{N}(X_n(j)-X_n(i)) \,. \qquad (5)$$

In the synchronized state, it is clear that the individual maps will be governed by $X_{n+1}(i)=F(X_n(i))+\kappa X_n(i)$. Written in this form, we see that whereas the individual map $F(X)$ may be chaotic, it is possible by modifying κ to make the new map, $G(X)=F(X)+\kappa X$, stable around the synchronized state. Thus the stability of $G(X)$ is intrinsically related to the observed synchronization. Consequently, future studies of synchronization may also benefit from examination of stable states in the isolated systems.

Finally, we can confirm that there is a very particular relation between the nonlinearity in component maps and synchronization: maps with stable windows—e.g., C^1 unimodal maps—can be used to synchronize by the mechanism described. Since many coupled systems can be reduced to such a map by a Poincaré surface of section, we anticipate that this synchronization mechanism may be common.

The author wishes to thank S. Strogatz for helpful discussions.

[1] K. Kaneko, Phys. Rev. Lett. **63**, 219 (1989); R. E. Amritkar and P. M. Gade, *ibid.* **70**, 3408 (1993); O. Tai, C. Skorupka, and L. M. Pecora, Phys. Lett. A **187**, 175 (1994).

[2] I. Aranson, D. Golomb, and H. Sompolinksy, Phys. Rev. Lett. **68**, 3495 (1992).

[3] P. Hadley, M. R. Beasley, and K. Wiesenfeld, Phys. Rev. B **38**, 8712 (1988); S. H. Strogatz, C. M. Marcus, R. E. Mirollo, and R. M. Westervelt, Physica D **36**, 23 (1989).

[4] Y. Kuramoto, *Chemical Oscillations, Waves, and Turbulence* (Springer, Berlin, 1984); Y. Kuramoto and I. Nishikawa, in *Cooperative Dynamics in Complex Physical Systems,* edited by H. Takayama (Springer, Berlin, 1989).

[5] J. Buck and E. Buck, Sci. Am. **234**, 74 (1976); M. K. McClintock, Nature **229**, 244 (1971); R. E. Mirollo and S. H. Strogatz, SIAM J. Appl. Math **50**, 1645 (1990); A. Sherman and J. Rinzel, Biophys. J. **59**, 547 (1991); A. T. Winfree, J. Theor. Biol. **16**, 15 (1967).

[6] In this Brief Report, we confine our analysis to synchroni-

zation to periodic states. Synchronization to nonperiodic states may also be possible.
[7] In this clustered period-3 condition, roughly a third of the maps are in each of the three periodic states at any point in time.
[8] Alternatively, one could modify the maps to include more complicated terms, leading to a variety of periodic windows. See, e.g., J. Ringland, N. Issa, and M. Schell, Phys. Rev. A **41**, 4223 (1990).

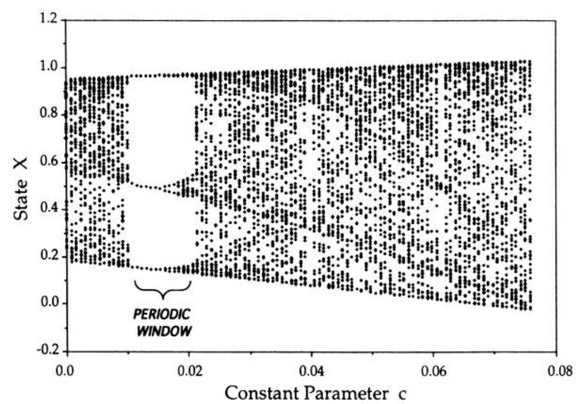

FIG. 1. Bifurcation diagram for Eq. (2).

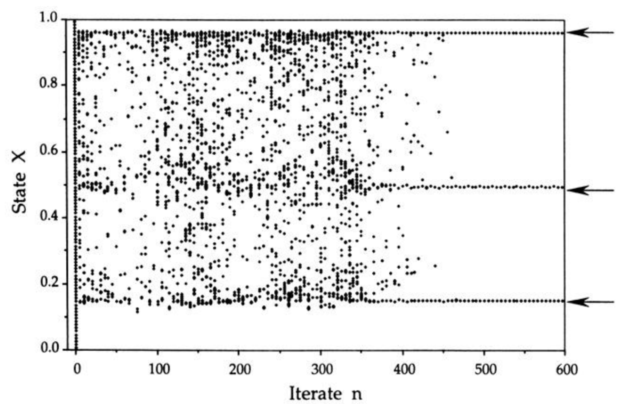

FIG. 2. Synchronization of 100 globally coupled logistic maps ($k = 0.023$ to hit $c = 0.015$); we mention that *all* map states are plotted.

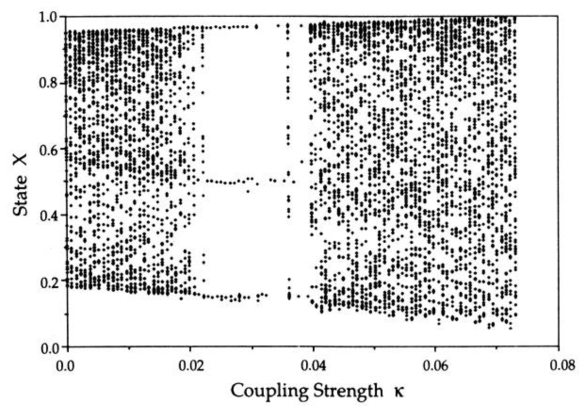

FIG. 3. Synchronization diagram for logistic map lattice.

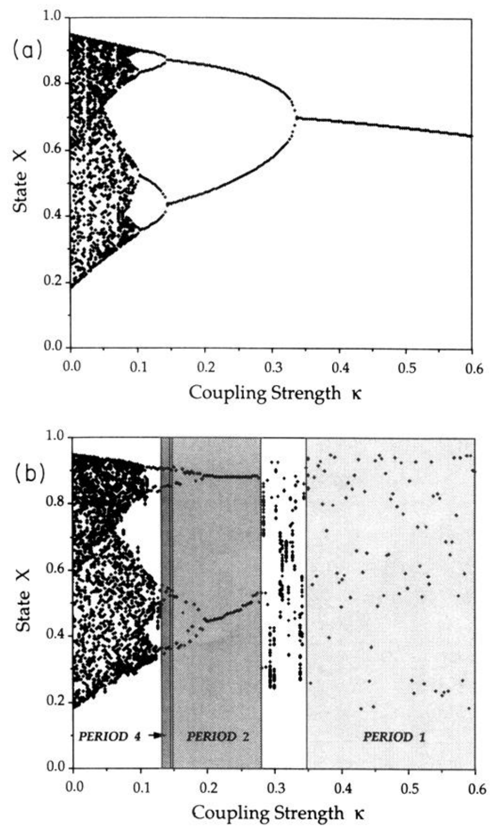

FIG. 4. (a) Bifurcation diagram for isolated logistic map, Eq. (4); (b) synchronization diagram for 100 coupled logistic maps, Eq. (3).